NOISE-MEDIATED COOPERATIVE BEHAVIOR IN A SYSTEM OF COUPLED DC SQUIDS

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ABSTRACT

We study the oscillator equations describing a system of coupled dc SQUIDs. The circulating current in each SQUID is inductively and globally coupled to the loop currents in the other SQUIDs. Just beyond the onset of spontaneous oscillations, the system shows significantly enhanced sensitivity to very weak magnetic fields. The ability to quantify the oscillation frequency permits its explotaition as a detection/analysis tool in remote sensing applications. Here we present quantitative results about such oscillation frequency and its scaling in terms of the control parameters, and the number of SQUIDs involved. For infinitely many coupled SQUIDs, the thermodynamic limit, we derive a nonlinear Fokker-Planck equation. This mean-field equation allows us to explore the various regimes of operation of the system analytically as well as numerically.

1. INTRODUCTION

The study of nonlinear dynamical behavior in systems that undergo bifurcations via changing a control parameter is of considerable interest [1]. When tuned near the onset of bifurcations, dynamical systems can display an enhanced sensitivity to external perturbations with the response characterized by signal amplification, often with a concommittant lowering of an environmental noise-floor, but also (depending on the parameters) potentially adverse effects e.g. the amplification of environmental fluctuations with an accompanying lowering of the response SNR. Among the nonlinear systems that have been studied in recent years, the dc SQUID has recently received considerable attention, since it is a device that is severly constrained by noisefloor issues and one in which a detailed study of the (noise-mediated) cooperative behavior in various regimes of operation can yield clever techniques for confronting noise-related performance issues that constrain current devices.

The dc SQUID [2] consists of two Josephson junctions symmetrically inserted into a superconducting loop, and characterized by a two-dimensional (2D) set of dynamical equations for the junction Schrödinger phase differences. Our interest in the SQUID stems from its relevance as the most sensitive detector of magnetic fields, being widely used in a variety of fields including biomagnetics, geophysics and explosive detection.

Past research activities have focused primarily on designing and developing sophisticated shielding and noise-cancellation techniques to render SQUIDs more noise-tolerant. Noise-mediated cooperative behavior has been studied by us in the dc SQUID. Recent calculations [3] showed that the application of a target or “injection locking” signal at (or slightly detuned from) the running frequency, resulted in a lowering of the noisefloor at all frequencies, and more recent calculations [4] have, in fact, revealed a true “resonance” behavior in the response: a maximum in the average loop screening current or its spectral amplitude occurs when the target signal frequency matches the deterministic running frequency.

In this paper, we study the dynamical behavior in a globally coupled ensemble of dc SQUIDs in the presence of background noise. There are likely several different ways to construct experimentally a system of globally coupled SQUIDs. One possible experimental scenario could be the following: a network of pickup coils is connected in parallel to sense and sum the fluxes of all the SQUIDs. Some of the resulting flux (depending of the coupling strength) is applied to each SQUID by feeding back the total output current through a feedback coil. Since every SQUID can interact magnetically with the next neighbor, leading to a local-type coupling, it is advisable to place each SQUID into a shielded environment. This form of coupling gives rise to a near-global coupling similar to the one we have proposed here, with each SQUID subject to a flux due to all the other SQUIDs in the same way. Global coupling is also most amenable (of all the possible coupling schemes) to theoretical treatment.

We study our globally coupled system both analytically and numerically, finding that the system exhibits static and oscillatory regimes of operation, completely analogous to the single SQUID case, studied earlier [5, 3]. Our analysis stems from the center manifold reduction technique that was applied to the single SQUID problem, and recently described semi-analytic techniques for solving the 2D FPE

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associated with the Langevin dynamics [6]. This previous work is conveniently generalized to treat the N-SQUID (N may be arbitrary) case with global coupling, but with *non*-identical SQUIDs.

The paper is organized as follows: After a rapid overview of the dc SQUID dynamics in section 2, we present an analytical calculation of the frequency of the running state and its scaling in terms of the distance from the bifurcation point. Finally (section 4) we investigate the effect of noise on the coupled system dynamics. Our results are summarized and discussed in section 5.

2. BACKGROUND AND MODEL EQUATIONS

In terms of the Schrödinger phase angles $\delta_{1,2}$ of the two (assumed identical) Josephson junctions we can express the measureable screening current I in the loop:

$$\beta \frac{I}{I_0} = \delta_1 - \delta_2 - 2\pi \frac{\Phi_e}{\Phi_0}, \quad (1)$$

where $\beta \equiv 2\pi L I_0/\Phi_0$ is the nonlinearity parameter, I_0 the junction critical current, L the loop inductance, and Φ_e an external applied magnetic flux, $\Phi_0 \equiv h/2e$ being the flux quantum. In the absence of noise and a target magnetic flux, we can use the RSJ model to write down equations for the currents in the two arms of the SQUID via a lumped circuit representation; when transformed via the Josephson relations $\dot{\delta}_i = 2eV_i/\hbar$ linking the voltage and the quantum phase difference across the junction i, these equations take the form,

$$\tau\dot{\delta}_1 = \frac{I_b}{2} - I - I_0 \sin\delta_1, \; \tau\dot{\delta}_2 = \frac{I_b}{2} + I - I_0 \sin\delta_2, \quad (2)$$

where $\tau \equiv \hbar/2eR$, R being the normal state resistance of the junctions. The two natural experimental control parameters are the applied dc magnetic flux Φ_e and the dc bias current I_b, which we take to be symmetrically applied to the loop. It is convenient to rescale time by τ and introduce a scaled flux $\Phi_{ex} \equiv \Phi_e/\Phi_0$ and bias current $J \equiv I_b/(2I_0)$.

This system exhibits two regimes of operation [5, 3]. For a fixed Φ_{ex}, a saddle-node connection takes place when the bias current J exceeds a critical value J_c. For $J < J_c$, the noiseless system has two fixed points, one stable (a node) and one unstable (a saddle). This is the "superconducting regime" with the potential energy function admitting of stable minima corresponding to a current conservation $2J = \sin\delta_1 + \sin\delta_2$. For $J > J_c$ the fixed points disappear and we obtain oscillatory solutions whose frequency obeys the characteristic square-root scaling law [1]. This latter regime is the so-called "running regime", and the properties of the solutions near the bifurcation have recently been studied [3]. We extend the model equations (2) to describe a system of globally linearly coupled dc SQUIDs, the theoretical variables of interest being the Schrödinger phase differences $\delta_j^{(k)}$ $(j = 1, 2)$ across each Josephson junction of the k^{th} SQUID (k=1,...,N):

$$\frac{\tau_k}{I_{0k}} \dot{\delta_j^{(k)}} = J_k + (-1)^j \frac{I_k}{I_{0k}} - \sin\delta_j^{(k)}, \quad (3)$$

where I_k represents the circulating current, J_k the normalized (to I_{0k}) externally applied bias current, I_{0k} the critical current of the junctions, and $\tau_k = \hbar/(2eR_k)$ is a characteristic time constant (R_k being the normal state resistance of the junctions). The circulating current I_k at the k*th* SQUID is induced in the loop by an external magnetic flux Φ_e which is assumed identical for all SQUIDs. Each SQUID is inductively coupled to the loop currents of the remaining SQUIDs with equal mutual inductance coupling of strength M. The circulating current can be written in the form:

$$\beta_k \frac{I_k}{I_{0k}} = \delta_1^{(k)} - \delta_2^{(k)} - \frac{2\pi}{\Phi_0}\left(\Phi_e + M \sum_{m\neq k} I_m\right) \quad (4)$$

where $\beta_k \equiv 2\pi L_k I_{0k}/\Phi_0$ is the nonlinearity parameter of the k^{th} SQUID, L_k being its loop inductance. Note that the circulating current I_m appearing on the right hand side of (4) is itself a function of every other circulating current. In fact, it represents an infinite nested series that in general cannot be summed in closed form. However, in such a case an expansion in powers of M can be obtained.

3. GLOBALLY COUPLED SQUIDS WITHOUT NOISE

As mentioned earlier, a single dc SQUID exhibits two different states of operation: a superconducting state where the long-time phases are time-independent, and a "running state" characterized by oscillatory phases. Quantifying small changes in the frequency that occur in the presence of external (target) signals could afford a detection mechanism, and experiments involving synchronization to an external signal or to another SQUID would inevitably benefit from *a priori* knowledge of the oscillation frequency in terms of the bias parameters.

Our calculation is made possible by the fact that close to the singular point there is a well-defined separation of time-scales that permits a center manifold reduction of the effective phase space. This renders the dynamics accesible to analytic computation close to the onset of the bifurcation. The complete calculations can be found in [4], here we show the main steps of such a calculation. In a first step a local analysis close to the singular point captures the very slow dynamics which is responsible for the long period of the running state. Secondly, we get a quantitative expression for the running period by solving the evolution equation on the center manifold. The result is increasingly accurate as $J \to J_{c1}$, J_{c1} being the critical bias current corresponding to the SQUID with smaller β. The evolution on the center manifold is given by

$$\dot{v}_1 = (J - J_{c1})\cos\theta_1 - \eta\sin\theta_1 + \alpha v_1^2 + 2\gamma\frac{\varepsilon^2}{\beta_1}v_1 + O((J - J_{c1})^3), \quad (5)$$

where θ_1, α, γ, and η depend on the bias parameters, and the critical fixed point (see Ref. [5] for calculational details), and $\varepsilon = 2\pi M/\Phi_0$. Integrating the last equation, we obtain the solution

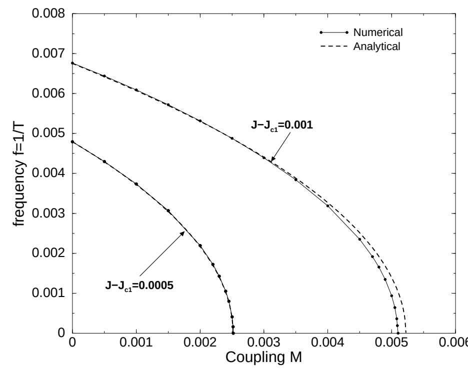
$$v_1(t) = \sqrt{\frac{F}{\alpha} - \frac{\gamma^2\varepsilon^4}{\alpha^2}}\tan\left(t\sqrt{F\alpha - \gamma^2\varepsilon^4}\right) - \frac{\gamma\varepsilon^2}{\alpha}, \quad (6)$$

where $F = (J - J_{c1})\cos\theta_1 - \eta\sin\theta_1$. Thus, for the frequency of the running state we find

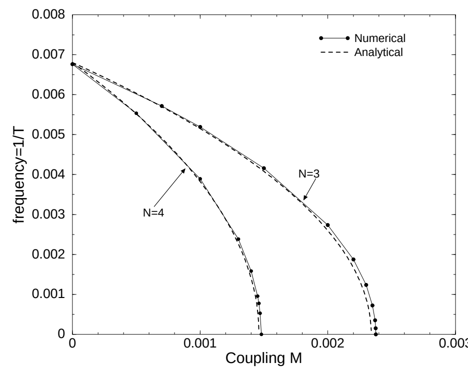
$$f = \sqrt{F\alpha - \gamma^2\varepsilon^4}/2\pi \quad (7)$$

Notice that the frequency of the running state decreases when the coupling (i.e. ε) increases. In fact, there exists a critical value of the coupling above which the oscillation frequency is zero: too strong a coupling "kills" the running states.

Fig. 1(a) shows a comparison between the numerical simulations of the system of equations (3) and the frequency obtained by using the analytical expression (7) for the case of $N = 2$. The agreement between the numerical results and the analytical results is excellent, particularly, as expected, for small values of $J - J_{c1}$. Fig. 1(b) shows a similar comparison for the case of $N = 3$, and $N = 4$.



(a)



(b)

Fig. 1. Comparison between the numerical simulations of the system of equations (3), and the frequency obtained analytically for (a) $N = 2$, and (b) $N = 3$, $N = 4$ SQUIDs. Two different values of the bias current are used in figure (a): $J = J_{c1} + 0.001$, and $J = J_{c1} + 0.0005$, with $J_{c1} = 0.821152$. Parameters are $\beta_1 = 0.9, \beta_2 = 1$, $\beta_3 = 1.1$, $\beta_4 = 1.3$, and $\Phi_{ex} = 0.2$.

4. GLOBALLY COUPLED SQUIDS WITH NOISE

In this section, we present some results involving the model equations (3) in presence of thermal noise, and in the limit of infinitely many SQUIDs. Specifically, we investigate the system (3)-(4), up to order M^2, in presence of Gaussian white noises, with $< \xi_i^{(j)}(t) >= 0, \quad < \xi_i^{(l)}(t)\xi_j^{(m)}(t') >= 2D\delta_{ij}\delta_{lm}\delta(t-t')$.

We are interested in the analytical investigation of the Langevin dynamics above for the case of very large N. A neat picture of such a case can be given by the limiting-model obtained when $N \to \infty$ (thermodynamic limit). In this limit, it is well known [7] that models with mean-field coupling are described by an evolution equation for one-particle probability density. Such a probability density $\rho(\delta_1, \delta_2, t)$ is asymptotically in the limit, $N \to \infty$, the solution of the following nonlinear Fokker-Planck equation (FPE):

$$\frac{\partial\rho}{\partial t} = D\left[\frac{\partial^2\rho}{\partial\delta_1^2} + \frac{\partial^2\rho}{\partial\delta_2^2}\right] - \frac{\partial}{\partial\delta_1}(v_1\,\rho) - \frac{\partial}{\partial\delta_2}(v_2\,\rho). \quad (8)$$

The drift-terms are given by

$$v_1(\delta_1,\delta_2,t) = J - \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_{ex} - \frac{2\pi\,M}{\Phi_0}\bar{I}) - \sin\delta_1 \quad (9)$$

$$v_2(\delta_1,\delta_2,t) = J + \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_{ex} - \frac{2\pi\,M}{\Phi_0}\bar{I}) - \sin\delta_2, \quad (10)$$

with n chosen in order to obtain a periodic continuation of the coefficients. J, β, and Φ_{ex} are picked up from a given distribution, $g(J)$, $f(\beta)$, and $h(\Phi_{ex})$ respectively, and the

the average screening current is now given by

$$\bar{I}(t) = \int dJ g(J) \int d\beta f(\beta) \int d\Phi_{ex} h(\Phi_{ex}) \int_0^{2\pi} \int_0^{2\pi} d\delta_1 d\delta_2 \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_{ex})\rho(\delta_1, \delta_2, t). \quad (11)$$

The probability density is required to be 2π-periodic as a function of δ_1, and δ_2, and normalized to one.

To study the nonlinear FPE we proposed a new numerical method, which consists in a generalization of a spectral method for a single SQUID already derived in [6]. The idea is to expand ρ in Fourier series, exploiting the 2π- periodicity in δ_1, and δ_2. Introducing such expansion into the Fokker-Planck equation, we obtain a infinite hierarchy of ordinary differential equations for the moments r_n^m, where the average screening current $\bar{I}$ is given by,

$$\bar{I}(t) = \frac{I_0}{\beta}\left[-8\pi^2 \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \mathrm{Im}(r_l^{-l} e^{-i\,2\pi\, l \Phi_{ex}})\right]. \quad (12)$$

The numerical method consists of truncating the hierarchy of first-order, coupled nonlinear differential equations, for a reasonable number of modes $N = -N, \ldots, N$, and $m = -M, \ldots, M$, setting $r_{N+1}^{M+1} = r_{-N-1}^{-M-1} = 0$. The number of modes N, M should be chosen so large than the numerical results do not depend on N, and M.

In Fig. 2, we show a comparison between the numerical solution obtained by means of the Fokker-Planck approach for a one-SQUID probability density, and the solution of the Langevin equations for a large number of SQUIDs ($N = 500$). It is remarkable that the solution of the FPE, corresponding to the limiting-model obtained when $N \to \infty$, is seen to provide excellent agreement with the finite size case. This manifests the fact that $N = 1000$ is already close to infinity for a practical purpose.

5. CONCLUSIONS

This work represents a continuation of a line of research into the behavior of dynamical systems in the vicinity of bifurcations. Our focus has been on large arrays of such oscillators, motivated by earlier work on the (potentially) positive role of coupling for a variety of signal processing applications. Clearly, increasing the coupling and/or the number of coupled entities should afford a valuable tool for lowering the spontaneous oscillation frequency. In turn, this makes it more convenient to detect low frequency oscillations in an external signal by matching them to the running frequency (which plays the role of an internal "clock"). When the target signal is applied to every SQUID, the enhancement of the spectral response is amplified; this is an extension of our work on single SQUID dynamics in the presence of a target signal and a noise-floor, and elucidates the role of coupling, as well as the necessity of choosing the coupling coefficients carefully.

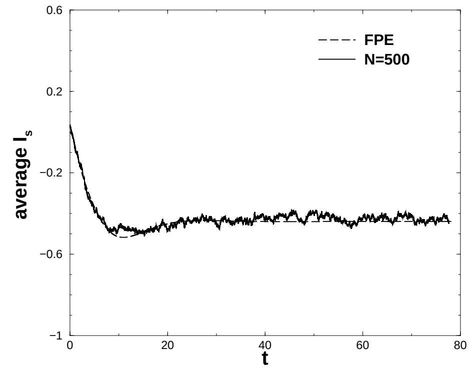


Fig. 2. Comparison between the numerical simulation of the Fokker-Planck equation (spectral method), and the Langevin equations with $N = 500$. SQUIDs are identical, and parameters are $J_0 = 0.35$, $\beta_0 = 1$, $\Phi_{ex_0} = 0.4$, $M = 0.01$, and $D = 0.1$.

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